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On the least action principle in cosmology

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ABSTRACT
Given the present distribution of mass tracing objects in an expanding universe, we develop and test a fast method for recovering their past orbits using the least action principle. In this method, termed FAM for Fast Action Minimization, the orbits are expanded in a set of orthogonal time-base functions satisfying the appropriate boundary conditions at the initial and final times. The conjugate gradient method is applied to locate the extremum of the action in the space of the expansion coefficients of the orbits. The TREECODE gravity solver routine is used for computing the gravitational fields appearing in the action and its gradients. The time integration of the Lagrangian is done using Gaussian quadratures. FAM allows us to increase the number of galaxies used in previous numerical action principle implementations by more than one order of magnitude. For example, orbits for the $\sim 15,000$ IRAS PSCz galaxies can be recovered in $\sim 12,000$ CPU seconds on a 400MHz DEC-Alpha machine. FAM can recover the present peculiar velocities of particles and the initial fluctuations field. It successfully recovers the flow field down to clusters scales, where deviations of the flow from the Zel'dovich solution are significant. We also show how to recover orbits from the present distribution of objects as in redshift space by direct minimization of a modified action, without iterating the solution between real and redshift spaces.

Key words: cosmology: theory – gravitation – dark matter – large scale structure of Universe

1 INTRODUCTION

In the standard cosmological paradigm, the present distribution of galaxies and their peculiar motions are the result of gravitational amplification of tiny initial density fluctuations. Accordingly, the mildly non-linear large scale structure observed today contains valuable information on the initial fluctuations. Given either the present galaxy distribution or the peculiar velocity field one can recover the growing mode of the initial density field (Peebles 1989, Weinberg 1992, Nusser & Dekel 1992, Gramman 1993, Giavalisco et al 1993, Croft & Gaztanaga 1998, Narayanan & Weinberg 1998). Methods for recovering the initial fluctuations can be very rewarding as one can directly address statistical properties of these fluctuations (Nusser & Dekel 1993). As has been shown by Nusser & Dekel (1992,1993) the initial density field recovered from peculiar velocity fields has a strong dependence on the value of the density parameter, Ω. On the other hand, a recovery from the galaxy distribution depends very weakly on Ω. Therefore by matching the statistical properties from the two recovered initial density fields one could provide an estimate for Ω. Gravitational instability theory, also provides tight relations between the present density and peculiar velocity fields (e.g., Nusser et al 1991). These relations have been an important tool in large scale structure studies, in particular for model independent estimates of the cosmological parameters. For example, a comparison of the measured peculiar velocities with those predicted from galaxy redshift surveys can yield Ω. (Davis, Nusser & Willick 1996, Willick et al 1996, da Costa et al 1998, Sigad et al 1998, Branchini et al 1999). These comparisons are done under an assumed form for the biasing relation between the mass and galaxy distribution. So deviations from the theoretical velocity-density relations may serve as an indication to the way galaxies trace the mass and hence to the interplay between galaxy formation and the large scale environment.

Most of the promising methods for recovering the initial growing mode and for relating the present peculiar velocity and density fields, are based on the least action principle (LAP). Peebles (1989) (hereafter P89) has pointed out that the equations of motion can be derived from the stationary variations of the action with respect to orbits subject to fixed final positions and vanishing initial peculiar velocities. P89 proposed minimizing the action with respect to the coefficients of an expansion of the orbits in terms of time-dependent functions satisfying the appropriate boundary conditions. Methods based on LAP are very powerful as the true orbits are recovered to any accuracy depending on the number of functions used in the expansion, at least in the laminar flow regime. They also provide simultaneous estimates of the present peculiar velocities and the initial fluctuations from a given distribution of galaxies.

So far, the action principle has been applied to study the dynamics of the Local Group of galaxies (P89, Peebles 1991, 1994, Dunn & Laflamme 1993, Branchini & Carlberg 1994) and to recover the peculiar velocity from the distribution of about 1100 galaxies within a redshift of 3000 km/s (Shaya, Peebles & Tully 1995). The application of LAP to larger data sets have been hindered by the heavy computational burden needed in current methods for minimizing the action. The LAP is the only technique with which one can probe the nonlinear behavior to any accuracy as opposed to Zel'dovich type approximations which break down near large mass concentrations like clusters of galaxies. Therefore, a fast method for implementing LAP can be rewarding. Such a method would enable us to apply LAP to large data sets like the various IRAS galaxy redshift surveys (1.2 Jy, Fisher et al 1995a, and PSCz, Saunders 1996) and portions of the Sloan Digital Sky Survey (Gunn & Knapp 1993). In contrast to previous numerical implementations of LAP, which use direct summation over inter-particle forces, we propose here an efficient method based on the TREECODE to compute gravitational forces and the potential energy. Also, our method expands the orbits in orthogonal time base functions and perform the time integration using the Gaussian quadratures method. All this increases the speed of the calculation by more than one order of magnitude over any previous implementation of LAP. We refer to our method by the acronym FAM for Fast Action Minimization.

The outline of the paper is as follows. In section 2 we review the least action principle and describe FAM. In section 3 we demonstrate the robustness of and show tests of the recovered peculiar velocities. In section 4 we describe extensions of the FAM to flux limited surveys, application from redshift surveys and biased distribution of galaxies. We conclude in section 5 with a discussion of our results and possible applications of FAM.

2 THE BOUNDARY VALUE PROBLEM AND THE LEAST ACTION PRINCIPLE

We follow the standard notation in which $a(t)$ is the scale factor, $H(t) = \dot{a}/a$ is the time-dependent Hubble factor, $\Omega = \rho_b/\rho_c$ is the ratio of the background density, ρ_b, of the universe to the critical density, $\rho_c = 3H^2/8\pi G$.

Assume that the underlying mass density field in a spherical volume V is sampled, in an unbiased way, by a discrete distribution of N galaxies (particles). In this sampling, if the average mass density in any cell of volume $\delta V << V$ is $\rho_{\delta V}$ then the number of particles in that cell is drawn from a Poisson distribution with mean $\bar{n}(\rho_{\delta V}/\rho_b)\delta V$, where $\bar{n} = N/V$ is the mean number density over the large volume V and we have assumed that the average mass density in V is ρ_b. Instead of using the usual time variable, t, we describe the evolution of the system in terms of the linear growing mode, $D(t)$, of density perturbations (e.g., Peebles 1980). Let $\boldsymbol{x}_i$ denote the comoving coordinate of the $i^{\rm th}$ particle and $\boldsymbol{\theta}_i = {\rm d}\boldsymbol{x}_i/{\rm d}D$ its velocity with respect to the time variable D. Neglecting interactions between matter interior and exterior to the volume V, system of particles obeys the the following Euler equations

$$\frac{{\rm d}\boldsymbol{\theta}_i}{{\rm d}D} + \frac{3}{2}\frac{1}{D}\boldsymbol{\theta}_i = \frac{3}{2}\frac{1}{D^2}\frac{\Omega}{f^2(\Omega)}\boldsymbol{g}(\boldsymbol{x}_i), \tag{1}$$

where $f(\Omega) = {\rm dln}D/{\rm dln}a \approx \Omega^{0.6}$ is the linear growth factor (e.g., Peebles 1980) and $\boldsymbol{g}$ represents the peculiar gravitational force field per unit mass. These equations are supplemented by the Poisson equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{g}(\boldsymbol{x}) = -\delta(\boldsymbol{x})\,, \tag{2}$$

which relates the divergence of $\boldsymbol{g}$ to the mass density contrast $\delta(\boldsymbol{x}) = \rho(\boldsymbol{x})/\rho_b - 1$ at any point $\boldsymbol{x}$ in comoving coordinate space. We can approximate δ from the discrete particle distribution by

$$\delta(\boldsymbol{x}) = \frac{1}{V}\sum_{i=1}^{N}\delta^{\rm D}(\boldsymbol{x}-\boldsymbol{x}_i) - 1\,, \tag{3}$$

where $\delta^{\rm D}$ is the Dirac delta function with unit integral over the volume V. Therefore, the field $\boldsymbol{g}$ is given by

$$\boldsymbol{g}(\boldsymbol{x}) = -\frac{1}{4\pi\bar{n}}\sum_i \frac{\boldsymbol{x}-\boldsymbol{x}_i}{|\boldsymbol{x}-\boldsymbol{x}_i|^3} + \frac{1}{3}\boldsymbol{x}. \tag{4}$$

Equations (1) and (4) constitute the equations of motion governing the evolution of the system of particles. We do not include the continuity equation in the equations of motion. The continuity equation is a constraint equation obeyed automatically as the particles move according to (1) and (4). The equations of motion involve second order time derivatives. Solving these equations can be seen either as an initial value problem or as a boundary value problem (Giavalisco et. al. 1993). Numerical N-body codes (e.g., Hockney & Eastwood 1981) are the usual tool for solving the relevant initial value problem where the positions and velocities of particles are specified at a given time. But, recovering the orbits of particles given their present positions is a boundary value problem where the solution must yield a homogeneous distribution of particles at the initial time $D = 0$. P89 stated the cosmological boundary value problem in the context of the least action principle. P89 also suggested an approximation to the orbits by minimization of the action with respect to a particular choice trial functions. In our notation, the action of the system of particles is

$$\mathrm{S} = \int_0^1 \mathrm{d}D \sum_i \left\{\frac{1}{2}D^{3/2}\boldsymbol{\theta}_i^2 + \frac{3}{2}\frac{1}{D^{1/2}}\frac{\Omega}{f^2(\Omega)}\left[\frac{1}{4\pi\bar{n}}\sum_{j<i}\frac{1}{|\boldsymbol{x}_i-\boldsymbol{x}_j|} + \frac{\boldsymbol{x}_i^2}{6}\right]\right\}, \tag{5}$$

where we have arbitrarily set $D = 1$ at the present time. Following P89 we minimize the action with respect to the coefficients of an expansion of the orbits by means of time dependent base functions $\{q_n(D), n = 1 \cdots n_{max}\}$. We write the position of each particle $\boldsymbol{x}_i(D)$ for $D < 1$ as

$$\boldsymbol{x}_i(D) = \boldsymbol{x}_{i,0} + \sum_{n=1}^{n_{max}} q_n(D)\boldsymbol{C}_{i,n}\,, \tag{6}$$

where $\boldsymbol{x}_{i,0}$ is the position of the particle at $D = 1$ and the vectors $\boldsymbol{C}_{i,n}$ are the expansion coefficients with respect to which the action is to be minimized. Since $\boldsymbol{x}_i(D = 1) = \boldsymbol{x}_{i,0}$, we choose the functions $q_n(D)$ such that $q_n(D = 1) = 0$. By taking derivatives of (6) with respect to D, we find that the velocity, $\boldsymbol{\theta}_i$, is

$$\boldsymbol{\theta}_i(D) = \sum_{n=1}^{n_{max}} p_n(D)\boldsymbol{C}_{i,n}\,, \tag{7}$$

where $p_n = \mathrm{d}q_n/\mathrm{d}D$. Stationary variations of the action with respect to $\boldsymbol{C}_i$ yield the following set of equations

$$\frac{\partial \mathrm{S}}{\partial \boldsymbol{C}_{i,n}} = \int_0^1 \mathrm{d}D\left[D^{3/2}p_n\boldsymbol{\theta}_i + \frac{3}{2}\frac{q_n}{D^{1/2}}\frac{\Omega}{f^2(\Omega)}\boldsymbol{g}_i\right] = 0\,. \tag{8}$$

If we integrate by parts the term involving the velocity in the previous equation, we arrive at

$$\left(D^{3/2}q_n\boldsymbol{\theta}_i\right)_{D=1} - \lim_{D\to 0}\left(D^{3/2}q_n\boldsymbol{\theta}_i\right) - \int_0^1 \mathrm{d}D D^{3/2}q_n(D)\left[\frac{\mathrm{d}\boldsymbol{\theta}_i}{\mathrm{d}D} + \frac{3}{2}\frac{1}{D}\boldsymbol{\theta}_i - \frac{3}{2}\frac{1}{D^2}\frac{\Omega}{f^2(\Omega)}\boldsymbol{g}_i\right] = 0\,. \tag{9}$$

Without the boundary terms on the left, these equations are equivalent to the equations of motion averaged over time with weight functions $D^{3/2}q_n$. The boundary terms are individually eliminated by the imposing the following two constraints on the time-base functions (Peebles 1989),

$$q_n(D=1) = 0 \qquad \text{and} \qquad \lim_{D\to 0} D^{3/2}q_n(D)\boldsymbol{\theta}(D) = 0\,. \tag{10}$$

2.1 The homogeneity condition and the time-base functions

We will work directly with the functions $p_n = \mathrm{d}q_n/\mathrm{d}D$ rather than q_n. The first constraint in (10) means that the positions of particles at $D = 1$ remain unchanged when varying $\boldsymbol{C}_{i,n}$. The second constraint is very flexible, it merely implies that $\mathrm{dln}p_n(D)/\mathrm{dln}D > -5/4$ for $D \ll 1$. However the functions p_n must lead to homogeneous initial particle distribution. A sufficient condition for the homogeneity is that the time dependence of the particle velocities near $D = 0$ matches that of the linear velocity growing mode which, with respect to the time D, is a constant. Orbits with velocities with initial time

dependence like that of the decaying mode ($\propto D^{-5/2}$), do not necessarily lead to homogeneous initial distributions, although the decaying mode is derived under the assumption of small perturbations. Therefore, initial homogeneity implies that one of the p_n must be a constant and the rest increasing functions of D. There are many functions satisfying our boundary conditions. Here we choose p_n to be linear combinations of $1, D, D^2 \cdots D^{n_{max}}$ (Giavalisco et. al. 1993) which satisfy the following orthonormality condition

$$\int_0^1 \mathrm{d}D D^{3/2} p_n(D) p_m(D) = \delta^{\mathrm{K}}_{m,n} \tag{11}$$

where δ^{K} is the Kronecker delta function. Orthonormality will prove useful in the numerical minimization of the action using the conjugate gradient method. The functions p_n can be constructed using the Gramm-Schmidt algorithm, however, in this case they can be derived from the expression

$$p_n(D) = A_n \frac{1}{D^{3/2}} \frac{\mathrm{d}^n}{\mathrm{d}D^n} \left[D^{3/2} D^n (1-D)^n \right], \tag{12}$$

where A_n are normalizing constant. With the orthonormality condition (11) the expression for the action gradients $\partial \mathrm{S}/\partial C_i^\alpha$ (8) becomes

$$\frac{\partial \mathrm{S}}{\partial \boldsymbol{C}_{i,n}} = \boldsymbol{C}_{i,n} + \frac{3}{2} \int_0^1 \frac{q_n}{D^{1/2}} \frac{\Omega}{f^2} \boldsymbol{g}_i(D) \, \mathrm{d}D \, . \tag{13}$$

2.2 The numerical action minimization

Given the orbit expansion (6) and the base-functions (12), the problem of recovering the orbits reduces to finding the value of the coefficients $\boldsymbol{C}_{i,n}$ where the action (5) has a minimum. Our method for minimizing the action, FAM, is based on the conjugate gradient method (CGM) (e.g., Press et. al. 1992). An efficient implementation of CGM calls for a fast way of computing the action and its gradients with respect to $\boldsymbol{C}$. Most of the computational cost comes from the potential energy term in the action and the gravitational force field, $\boldsymbol{g}$, in the gradients (13). These quantities involve summation over pairs and they have to be computed various times, in each step taken by CGM, for an accurate estimate of their integrals. We can achieve a significant improvement over previous schemes for minimizing the action if instead of computing the gravitational forces by direct summation, we use the TREECODE technique (e.g., Bouchet & Hernquist 1988). Although any of the fast techniques like the particle-mesh (PM) or particle-particle particle-mesh ($\mathrm{P^3M}$) (Hockney & Eastwood 1981) or the adaptive $\mathrm{P^3M}$ (Couchman 1991) can be used, the TREECODE is particularly suitable for our purposes since it can readily be implemented for particle distributions in a spherical region as in whole sky galaxy catalogs. We reduce even further the required number of gravitational field calculation by performing the time integration in the expression for the action and its gradients using the Gaussian quadrature scheme with $D^{-1/2}$ weights (Giavalisco et al 1993).

The boundary value problem is almost certain to have more than one solution (P89, Giavalisco et. al. 1993). Part of the reason for this is that one of the boundary conditions only prescribes time dependence of the velocities near the initial time $D = 0$ and does not specify their amplitude. This is not sufficient for a unique solution, in the presence of orbit mixing regions. So the action can have many minima corresponding to different solutions of the time averaged equations of motion. Therefore, we expect the minimum found by CGM to depend on the initial guess. However, our purpose is to recover motions on large scales which means that out of all minima we would like to find the one corresponding to orbits which do not deviate significantly from the Zel'dovich straight line approximation. A reasonable choice for the initial guess is then: $\boldsymbol{C}_{i,n} = 0$ for $n > 1$, and $\boldsymbol{C}_{i,1}$ obtained from (13) by substituting $\boldsymbol{g}_i(D) = D\boldsymbol{g}_{i,0}$ where $\boldsymbol{g}_{i,0}$ is the gravitational force field at $D = 1$. This means that the initial guess for the orbit of each particle is a motion in straight line with velocity given by the gravitational force field according to linear theory.

2.3 The dependence on H_0 and Ω

We would like to clarify here the dependence of the equations of motion and the recovered orbits on the present value of Hubble factor, H_0, and the density parameter, Ω. Since D is a dimensionless variable, the spatial coordinate $\boldsymbol{x}$ and the velocity $\boldsymbol{\theta} = \mathrm{d}\boldsymbol{x}/\mathrm{d}D$ have the same units, e.g., Mpc. We could also work in km/s if we define $H_0\boldsymbol{x}$ to be the spatial coordinate. Neither the action nor its gradients (13) involve the Hubble factor, so the recovered orbits are completely independent of the units with which we choose to express the orbits.

Figure 1. Recovered orbits from the FAM. Filled dots show present time positions for a random selection of N-body particles contained within a slice of thickens 10 h^{-1} Mpc. The solid lines represent their projected orbits. The upper plot is an enlargement of the central region in the lower plot.

The equations of motion expressed in terms of the time variable, D, are almost independent of Ω and the cosmological constant (e.g., Weinberg & Gunn 1990, Nusser & Colberg 1997, Mancinelli & Yahil 1995, Gramman 1993). This is because of the weak dependence of $\Omega/f^2 \approx \Omega^{-0.2}$ on the cosmological parameters in 1. However, as we shall see in subsection 4.2, the orbits recovered from the distribution of galaxies in redshift space rather in real space, will have a non-negligible dependence on Ω. One could then obtain orbits in a $\Omega \neq 1$ universe by appropriate scaling those recovered assuming flat universe.

3 TESTING FAM WITH AN N-BODY SIMULATION

FAM involves a number of parameters. These include the force softening scale in the TREECODE gravity solver, the number of time-base functions, the convergence tolerance parameter in CGM, (see Press et. al. 1992 for details). In addition, we have the initial guess $\boldsymbol{C}_{i,n}$ required by CGM. We resort to an N-body simulation to demonstrate the robustness of the method to these parameters and the initial guess. We use a high resolution simulation (Cole et. al. 1998) of cold dark matter in a flat universe with a cosmological constant. The matter density parameter at the final output of the simulation is $\Omega_0 = 0.3$ The simulation contained 192^3 particles in a periodic cube of side 345.6 h^{-1} Mpc. The simulation is normalized such that at the final output the linearly extrapolated *rms* value of the density fluctuations in a sphere of 8h^{-1}Mpc is $\sigma_8 = 1.13$. The simulation was run using a modified version of Couchman (1991) AP^{3}M N-body code with a force softening parameter of 0.27 h^{-1} Mpc. We test FAM using the distribution of $1.5(\cdot 10^4)$ particles selected at random from a spherical region of radius 80 h^{-1} Mpc in the simulation. We will refer to a standard FAM recovery as the one in which the orbits are expanded in six time-base functions given by (12), the tolerance parameter in CGM is 10^{-4}, and the initial guess for $\boldsymbol{C}_{i,n}$ is given

Figure 2. Robustness tests of FAM. *To the left*: each panel shows velocities (one component) ' recovered with $\boldsymbol{C}_{i,n} = 0$ as the initial guess *vs* those recovered with standard FAM. The panels correspond, respectively, to different values for the force softening parameter. *To the right*: each panel compares velocities recovered with two values of the convergence tolerance parameter as indicated at the top.

from linear theory, as described at the end of subsection (2.2). The total number of free coefficients $\boldsymbol{C}_{i,n}$ in standard FAM is $3 \times 1.5(\cdot 10^4) \times 6 = 2.7(\cdot 10^5)$, including the 3 spatial components.

We will focus on the performance of FAM at recovering the particle velocities at the final output of the simulation. However, it is instructive to visually examine the recovered orbits. The solid lines in figure 1 are two dimensional streamlines of particles contained, at the final time, in a slice of thickness of $10h^{-1}$ Mpc. The dots indicate the present positions of the particles. The streamlines shown in the plot correspond to orbits recovered with standard FAM and a force softening parameter of $0.5h^{-1}$Mpc. The deviations of the streamlines from straight lines are significant, especially in high density regions. These deviations are an indication of the failure of the Zel'dovich approximation.

We first assess the robustness of the recovered velocities against changes in the initial guess for $\boldsymbol{C}_{i,n}$. To do that we have compared the solution of standard FAM with the solution obtained using $\boldsymbol{C}_{i,n} = 0$ as the initial guess. The three panels on the left hand side in figure 2 compare between the FAM recovered velocities in the two cases[⋆]. The three panels show, respectively, results for three values of the force softening parameter, as indicated in each panel. Changing the initial guess did not introduce any systematic differences in the recovered velocities for all three values of the softening parameter. The scatter, although not negligible, is small compared to the scatter between the recovered true velocities (see figure 4 below). The right hand side of figure 2 compares the standard FAM velocities with those recovered with a convergence tolerance parameter of 10^{-5}. The correlation between the two velocities is very tight and no systematic differences are detected.

Having established the robustness of FAM we now proceed to check how well it reproduces true velocities of particles in the simulation. In figure 3 we show the scatter plots of recovered *vs* true velocities of randomly chosen particles. In

[⋆] In this and the following figures we plot comoving peculiar velocities $\boldsymbol{V} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t = H_0 f(\Omega_0)\boldsymbol{\theta}$ measured in units of km/s

Figure 3. Recovered *vs* true velocities for random particles in the simulation. Left, middle and right columns correspond, respectively, to recovery from, standard FAM, Zel'dovich approximation, and linear theory. Shown are velocities smoothed with a top-hat window of radius 5 (top row), 3 (middle), and 1 h^{-1} Mpc (bottom). Displayed in each panel, the parameters of the linear regression of recovered on true velocities. The 45° solid lines are drawn to guide the eye.

addition to velocities recovered with standard FAM (left column) we show results from the Zel'dovich approximation (middle column) and linear theory (right column). Velocities in the Zel'dovich approximation were obtained by running FAM with $n_{max} = 1$, i.e, with straight line orbits of the form $\boldsymbol{x}_i(D) = \boldsymbol{x}_{0,i} + D\boldsymbol{C}_{i,1}$. The Zel'dovich velocities should coincide with those which would have been recovered by the PIZA method of Croft & Gaztanaga (1998). The linear theory velocities are simply $H_0 f(\Omega_0)\boldsymbol{g}_{0,i}$ where $\boldsymbol{g}_{0,i}$ is the gravitational force field obtained from the particle distribution at the final time. In all cases the softening parameter is $0.5h^{-1}$Mpc in the computation of the gravity field. The recovered and true velocities in each panel with a top-hat window of the same length. The three rows show results for three smoothing lengths, as indicated in the figure. The top-hat smoothing replaces the velocity of each particle by the mean velocity of its neighbors within a distance equal to the smoothing width. The parameters of the best linear fit are shown on the top left corner of each plot. Linear theory performs very poorly, even when smoothing on scales as large as 5 h^{-1}. Evidently, the Zel'dovich approximation is a significant improvement over linear theory. Yet, a close visual inspection reveals a systematic bias in the Zel'dovich velocities. This is confirmed quantitatively by the parameters of the linear regression. The FAM velocities are almost unbiased even for the smallest smoothing width. Note also the small scatter of 150 - 250 km/s between the FAM and true velocities.

The superiority of FAM over the Zel'dovich approximation can be further appreciated from figure 4 where we plot the unsmoothed velocities. For both in FAM and Zel'dovich we we show results with force softening of 0.25 h^{-1} Mpc (top panels) and 3 h^{-1} Mpc (bottom panels) in the TREECODE. The lower softening matches the the one used for the force calculation in the original N–body simulation. The Zel'dovich solution fails to reproduce nonlinear motions for both values of the softening parameter. It typically underestimates the true velocities. FAM, however, provides unbiased estimates of the true velocities if the force softening is close to the one of the N-body simulation. FAM cannot model accurately the structure of the orbits on scales that the force softening length. This is the cause of the bias in FAM velocities recovered with the high value of the

Figure 4. FAM *vs.* Zel'dovich. Plotted are unsmoothed velocities. *To the right*: Zel'dovich *vs.* True. *To the left*: LAP *vs.* true. Shown are velocities recovered with softening parameters of 0.25 h^{-1} Mpc (top plots) and 3 h^{-1} Mpc (bottom). The parameters of the linear fit are shown in each panel

softening parameter (bottom right panel). The random errors in the unsmoothed FAM velocities are large ($\sim$ 500km/s) but they go down by a factor of 2 when smoothing on a scale of 1 h^{-1} Mpc (see bottom left panel in figure 3).

4 EXTENSIONS OF FAM

So far we had in mind an ideal situation in which we have a perfect volume limited unbiased distribution of galaxies in real space. All sky surveys like the IRAS sample, provide us with the redshift coordinates of galaxies. They are typically flux limited so that the observed number density of galaxies is a decreasing function of distance. Galaxies may also be biased tracers of the underlying mass density field. We will now outline how FAM can deal with these issues.

4.1 Selection effects and shot-noise

Suppose that a flux limited catalog is obtained from a parent volume limited catalog. If the observed number density in real space in a flux limited catalog is $n_0(\boldsymbol{x})$, then the corresponding number density in the volume limited catalog is $n_0(\boldsymbol{x})/\phi(|\boldsymbol{x}|)$, ϕ is the selection function. The gravitational field $\boldsymbol{g}$ can then be approximated by

$$\boldsymbol{g}(\boldsymbol{x}) = -\frac{1}{4\pi n_1}\sum_i \frac{1}{\phi_i}\frac{\boldsymbol{x}-\boldsymbol{x}_i}{|\boldsymbol{x}-\boldsymbol{x}_i|^3} + \frac{1}{3}\boldsymbol{x}, \tag{14}$$

where $\phi_i = \phi(\boldsymbol{x}_{i,0})$ and n_1 is an estimate of the number density in the volume limited catalog, for example, $n_1 \approx \sum_i \phi_i^{-1}/V$ where the sum is over galaxies within a spherical region of volume V around the observer. The potential energy term in the expression for action is also modified accordingly. So for flux limited surveys the gravitational potential and force fields are computed by the TREECODE technique with each particle having a mass proportional to the inverse of the selection function at its final position. We now examine the error in the recovered velocities and positions of particles as a result of the discrete sampling of the mass density field. We define the error covariance matrix between two quantities X and Y as $< \Delta X\, \Delta Y >$ where the symbol $< . >$ denotes averaging over many different discrete samplings of the underlying mass field with the same selection function and number of particles as the original flux limited distribution. The Δ denotes the difference between the true and recovered values. The true value being obtained by application of LAP on a sampling with an *infinite* number of particles, but with the same selection function as in the dilute distribution. We focus on computing the velocity error covariance matrix. The calculation for the recovered positions is similar. Using (7), The velocity error covariance matrix between two particles i and j can be written in terms of the coefficients $\boldsymbol{C}$ error matrix,

$$< \Delta\boldsymbol{\theta}_i\, \Delta\boldsymbol{\theta}_j >= \sum_{m,n} p_n p_m < \Delta\boldsymbol{C}_{i,n}\, \Delta\boldsymbol{C}_{j,m} > , \tag{15}$$

By setting to zero the action gradients (13) we find

$$< \Delta\boldsymbol{C}_{i,n}\, \Delta\boldsymbol{C}_{j,m} >= \frac{9}{4}\int_0^1 \mathrm{d}D\mathrm{d}D' \frac{q_n q'_m}{(DD')^{1/2}} < \Delta\boldsymbol{g}_i\, \Delta\boldsymbol{g}'_j > , \tag{16}$$

were quantities with and without the prime symbol are evaluated at times D' and D, respectively. This expression involves the gravity error covariance matrix computed between errors at different times. We can estimate this matrix under the assumption that the deviations from the Zel'dovich solution do not affect on its value. According to the Zel'dovich approximation, the gravity force acting on a particle at a time D is $\boldsymbol{g}_i = D(f^2/\Omega)\boldsymbol{g}_{i,0}$, where $\boldsymbol{g}_{i,0}$ is computed at the final time. With this approximation (16) becomes

$$< \Delta\boldsymbol{C}_{i,n}\, \Delta\boldsymbol{C}_{j,m} >= B_{n,m} < \Delta\boldsymbol{g}_{i,0}\, \Delta\boldsymbol{g}_{j,0} > , \tag{17}$$

where $B_{n,m} = (9/4)\int_0^1 q_n q'_m (DD')^{1/2}(ff')^2/(\Omega\Omega')\mathrm{d}D\mathrm{d}D'$. The calculation of the gravity error covariance at the final time is straightforward (Yahil et. al. 1991) and the result is

$$< \Delta\boldsymbol{g}_{i,0}\, \Delta\boldsymbol{g}_{j,0} >= \frac{1}{(4\pi n_1)^{1/2}} \sum_k \frac{1}{\phi_k^2} \frac{(\boldsymbol{x}_i - \boldsymbol{x}_k)(\boldsymbol{x}_j - \boldsymbol{x}_k)}{|\boldsymbol{x}_i - \boldsymbol{x}_k|^3 |\boldsymbol{x}_j - \boldsymbol{x}_k|^3} \tag{18}$$

4.2 Redshift space distributions

Typically galaxy surveys provide redshifts and angular positions of galaxies in the sky. Redshifts of galaxies differ from their distances as a result of peculiar velocities along the line of sight. This causes differences between the distribution of galaxies in real and redshift space (e.g., Kaiser 1987, Hamilton 1993). These differences are referred to as redshift distortions. Method for recovering the velocity from the redshift space distribution of galaxies in the nonlinear regime have so far relied on iterations between redshift and real space (Shaya et al 1995, Yahil et. al. 1991, Schmoldt and Saha, 1998). Here we show that FAM can be extend to treat redshift distortions by direct minimization of a modified action, without the use iterations. For simplicity, we restrict the analysis to volume limited surveys. We define the comoving *redshift* coordinate, $\boldsymbol{s}_0$, of a galaxy at the present time, as

$$\boldsymbol{s}_0 = H_0\boldsymbol{x}_0 + (\boldsymbol{V}_0 \cdot \hat{\boldsymbol{s}}_0)\hat{\boldsymbol{s}}_0 \tag{19}$$

the 0 subscript refers to quantities at the present time and the unit vector $\hat{\boldsymbol{s}}_0$ points in the direction of the line of sight to the galaxy. The comoving peculiar velocity of the galaxy is $\boldsymbol{V} = (\mathrm{d}\boldsymbol{x}/\mathrm{d}t) = DHf(\Omega)\boldsymbol{\theta}$.

The redshifts of galaxies are given. So the expansion of orbits in terms of time-dependent functions has to be such that the redshift coordinates as given by (19) are fixed under variations of the expansion coefficients. Defining parallel ($^{\parallel}$) and perpendicular ($^{\perp}$) directions to the line of sight at the present time, we write the expansion of the orbits as

$$\boldsymbol{x}^{\parallel}{}_i(D) \quad = \quad H_0^{-1}\boldsymbol{s}_{0,i} + \sum_n q_n(D)\boldsymbol{C}^{\parallel}{}_{i,n} - f_0 \sum_n p_{0,n}\boldsymbol{C}^{\parallel}{}_{i,n}$$

$$
\begin{aligned}
&= \boldsymbol{s}_{0,i} + \sum_n Q_n(D)\boldsymbol{C}^{\parallel}{}_{i,n} \\
\boldsymbol{x}^{\perp}{}_i(D) &= \sum_n q_n(D)\boldsymbol{C}^{\perp}{}_{i,n}
\end{aligned}
\tag{20}
$$

where $p_{0,n} = p_n(D=1)$ and $Q_n(D) = q_n(D) - f_0 p_{0,n}$. This expansion of the orbits satisfies the boundary conditions of fixed redshifts and angular positions on the sky. The role of the Hubble constant H_0 is adjusting the units. The trivial dependence on H_0 can be completely eliminated by working with $H_0\boldsymbol{x}$ instead of $\boldsymbol{x}$. Stationary first variations of the action (5) with respect to $\boldsymbol{C}^{\perp}$ subject to the boundary conditions (10) yield

$$
\int_0^1 \mathrm{d}D D^{3/2} q_n \left[\frac{\mathrm{d}\boldsymbol{\theta}_i^{\perp}}{\mathrm{d}D} + \frac{3}{2}\frac{\boldsymbol{\theta}_i^{\perp}}{D} - \frac{3}{2}\frac{1}{D^2}\frac{\Omega}{f^2}\boldsymbol{g}_i^{\perp}\right] = 0 \tag{21}
$$

which is the time averaged equation of motion of a particle in the plane perpendicular to its sight-line at the present time. On the other hand, stationary variations with respect to $\boldsymbol{C}^{\parallel}$ yield

$$
Q_{0,n}\boldsymbol{\theta}_{0,i}^{\parallel} - \int_0^1 \mathrm{d}D D^{3/2} Q_n \left[\frac{\mathrm{d}\boldsymbol{\theta}_i^{\parallel}}{\mathrm{d}D} + \frac{3}{2}\frac{\boldsymbol{\theta}_i^{\parallel}}{D} - \frac{3}{2}\frac{1}{D^2}\frac{\Omega}{f^2}\boldsymbol{g}_i^{\parallel}\right] = 0 \tag{22}
$$

where $Q_{0,n} = Q_n(D=1) = -f_0 p_{0,n}$ and $\boldsymbol{\theta}_{0,i}^{\parallel} = \sum p_{0,n}\boldsymbol{C}^{\parallel}$. These equations differ from the time averaged equations of motion by a boundary term. This term can be eliminated by adding to the action (5) a kinetic energy term corresponding to the line of sight parallel degree of freedom, as follows

$$
\mathcal{S} = \mathrm{S} + \frac{1}{2}\sum_i \left(\boldsymbol{\theta}_{0,i}^{\parallel}\right)^2 . \tag{23}
$$

Minimization of the the modified action readily yields the orbits expansion parameters $\boldsymbol{C}_{i,n}$ (see Schmoldt and Saha 1998, for a similar treatment of this problem).

The recovered orbits from redshift space depend on Ω through the f_0 in the expansion (20). The effect of this dependence in the linear regime is elucidated in Nusser & Davis (1994). Since linear theory overestimates the true velocities (Nusser et. al. 1991), this Ω dependence will be weaker in the nonlinear regime.

4.3 Biased distributions

So far we have assumed that the number of galaxies in a small cell is proportional the the average mass density in the cell. However, galaxies are most likely to be biased tracers of the mass distribution as indicated by the relative bias between galaxies of different luminosities and morphological types (e.g, Loveday et al 1995). For simplicity of notations we discuss here how our scheme can incorporate biasing only for volume limited galaxy distributions in real space. Suppose we are given smooth versions of the galaxy number density and the mass density fields. Working with smoothed fields seems reasonable because galaxy formation at a given point is likely to be affected by the nearby dark matter environment. We also assume that the smoothing scale is large enough such that the smoothed galaxy number density field is not contaminated by shot-noise. Let δ^g and δ be, respectively, the galaxy number density contrast and the mass density contrast, both smoothed with the same smoothing window of width fixed by the physical processes involved in galaxy formation. For unbiased galaxy distribution $\delta^g = \delta$ and for the familiar linear biasing $\delta^g = b\delta$ where b is the linear bias factor. Here δ^g is a nonlinear function of δ, which we assume to be local and deterministic (e.g., Dekel & Lahav 1998). We characterize the biasing relation at any point in space by the ratio

$$
\mathcal{W} \equiv \frac{1+\delta}{1+\delta^g}.
$$

Our definition of biasing in terms of smooth fields inevitably implies that the galaxy distribution does not contain information on the structure of mass density on scales smaller than the smoothing scale length. Given $\mathcal{W}$ we define the following density field

$$
\varrho(\boldsymbol{x}) = \frac{\rho_b}{\bar{n}}\mathcal{W}(\boldsymbol{x})\frac{1}{V}\sum_i \delta^{\mathrm{D}}\left(\boldsymbol{x} - \boldsymbol{x}_i\right) . \tag{24}
$$

This field serves as an unbiased estimate of smoothed underlying mass distribution. The gravitational force field, $\boldsymbol{g}$, is then estimated from ϱ by

$$\boldsymbol{g}(\boldsymbol{x}) = -\frac{1}{4\pi\bar{n}}\sum_i \mathcal{W}_i \frac{\boldsymbol{x}-\boldsymbol{x}_i}{|\boldsymbol{x}-\boldsymbol{x}_i|^3} + \frac{1}{3}\boldsymbol{x}, \tag{25}$$

where $\mathcal{W}_i = \mathcal{W}[\boldsymbol{x}_i(D)]$. If the galaxies and the mass particles share the same velocity field then the continuity equation implies that $\mathcal{W}$ remains constant along the streamlines so that $\mathcal{W}[\boldsymbol{x}_i(D)] = \mathcal{W}(\boldsymbol{x}_{i,0})$ (e.g., Nusser & Davis 1994, Fry & Gaztanaga 1995). Therefore, it is sufficient to specify the biasing relation at the present time. The net effect of biasing is that it changes the weight assigned to each particle in the calculation of gravitational fields. This can readily be incorporated in the TREECODE by assigning to each particle a mass proportional to $W(\boldsymbol{x}_{i,0})$.

5 DISCUSSION

We have presented a fast method for solving the boundary value problem of recovering the orbits of particles from their present positions assuming homogeneous initial conditions. The method which we term FAM is based on P89 implementation of lest action principle in a cosmological contest. It can be applied to distribution of galaxies in redshift space. It can also very easily incorporate any local biasing relation. FAM is suitable for recovering orbits from large galaxy redshift surveys such as the PSCz. It can also be applied to large portions of the future Sloan Digital Sky Survey.

We have described the method assuming that the number of expansion coefficients is the same for all galaxies. However, this need not be the case. For example, galaxies in low density regions can be assumed to move along straight line just like in the Zel'dovich approximation. This can significantly speed up FAM especially for the flux limited surveys with dilute galaxy distribution at large distances from the observer.

We have used an N-body simulation to show that FAM recovers very well the final velocities from a given volume limited particle distribution in real space. However, galaxy surveys provide galaxy distribution in redshift space. Redshift distortions introduce the nuisance of multi-valued zones where particles overlap in redshift space while they are far apart in real space. FAM allows a recovery of the orbits non-iteratively from redshift space data by direct minimization of a modified action. We believe that this should mitigate the effects of multi-values zones in the recovered orbits. Tests of FAM recovery from both flux and volume limited distributions in redshift space are underway.

In this work we concentrated on how well FAM can reconstruct objects' peculiar velocities and we did not check how well it can recover initial fluctuations. Judged by its superiority over the Zel'dovich solution at recovering the present velocities, FAM is expected to perform well at recovering the density fluctuation at any time in the past. We have outlined FAM can incorporate possible biasing between the mass and galaxies. We have shown that we only need to specify a biasing relation at the present time. The recovery of the orbits is sensitive to the biasing relation. So one can tune the biasing relation so that the clustering amplitude of the recovered mass density field varies with time according to hierarchical structure formation. In hierarchical clustering, the evolution of clustering amplitude, as measured for example by the correlation function, is almost independent of the linear mass power spectrum (e.g., Hamilton et. al. 1991, Jain, Mo & White 1995, Peacock & Dodds 1995), and of the cosmological parameters if expressed as a function of the linear density growing mode, D (e.g., Nusser & Colberg 1997). Therefore, one can determine the biasing relation independent of the cosmological parameters and the details of the dark matter model.

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